

# Mathematical Induction

Section 5.1

# Section Summary

- Mathematical Induction
- Examples of Proof by Mathematical Induction
- Mistaken Proofs by Mathematical Induction
- Guidelines for Proofs by Mathematical Induction

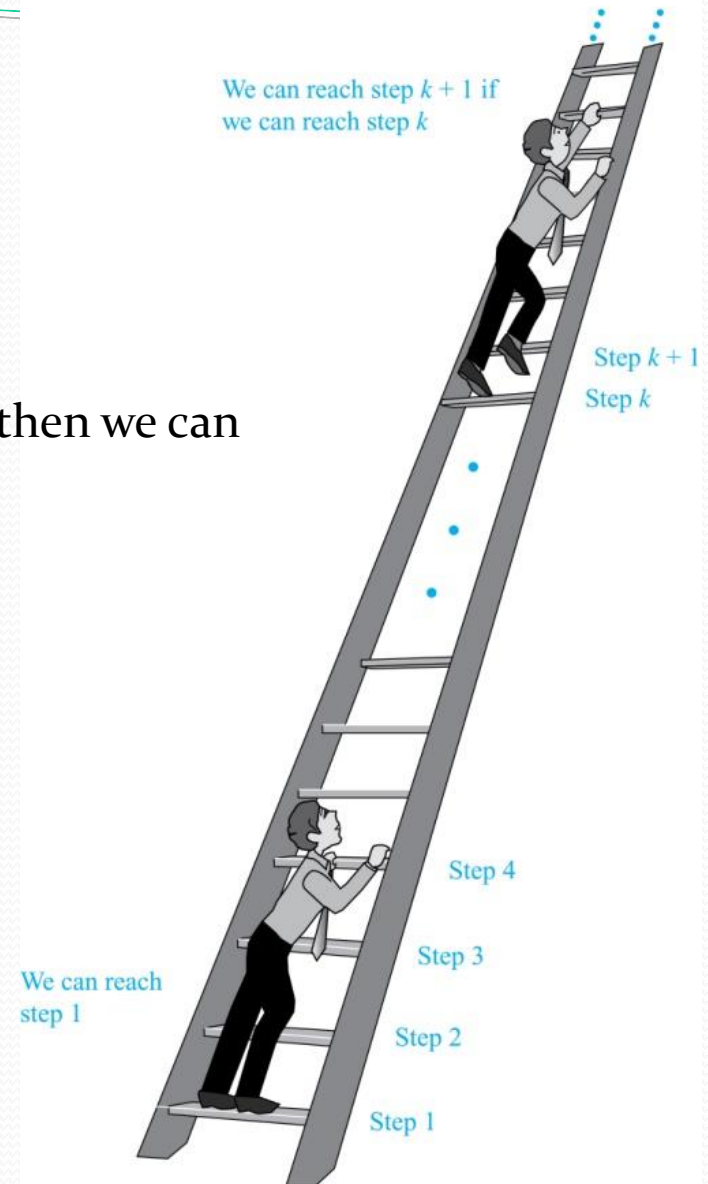
# Climbing an Infinite Ladder

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

This example motivates proof by mathematical induction.



# Principle of Mathematical Induction

*Principle of Mathematical Induction:* To prove that  $P(n)$  is true for all positive integers  $n$ , we complete these steps:

- *Basis Step:* Show that  $P(1)$  is true.
- *Inductive Step:* Show that  $P(k) \rightarrow P(k + 1)$  is true for all positive integers  $k$ .

To complete the inductive step, assuming the *inductive hypothesis* that  $P(k)$  holds for an arbitrary integer  $k$ , show that  $P(k + 1)$  must be true.

## **Climbing an Infinite Ladder Example:**

- **BASIS STEP:** By (1), we can reach rung 1.
- **INDUCTIVE STEP:** Assume the inductive hypothesis that we can reach rung  $k$ . Then by (2), we can reach rung  $k + 1$ .

Hence,  $P(k) \rightarrow P(k + 1)$  is true for all positive integers  $k$ . We can reach every rung on the ladder.



# Important Points About Using Mathematical Induction

- Mathematical induction can be expressed as the rule of inference

$$(P(1) \wedge \forall k (P(k) \rightarrow P(k + 1))) \rightarrow \forall n P(n),$$

where the domain is the set of positive integers.

- In a proof by mathematical induction, we don't assume that  $P(k)$  is true for all positive integers! We show that if we assume that  $P(k)$  is true, then  $P(k + 1)$  must also be true.
- Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point  $b$  where  $b$  is an integer. We will see examples of this soon.

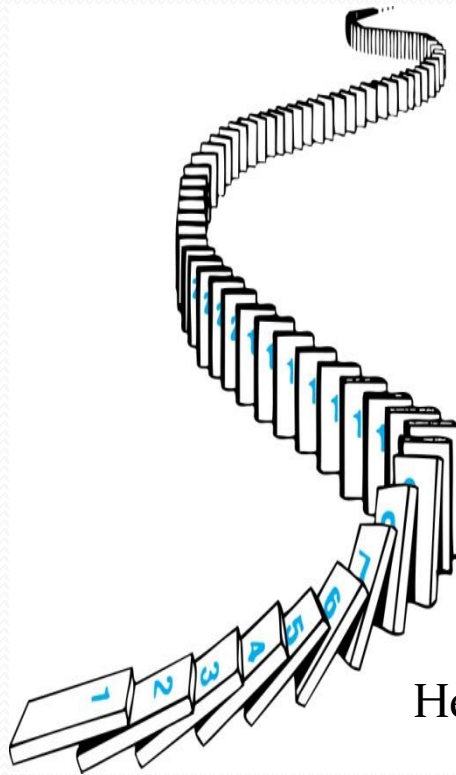
# Validity of Mathematical Induction

- Mathematical induction is valid because of the well ordering property, which states that every nonempty subset of the set of positive integers has a least element (see Section 5.2 and Appendix 1). Here is the proof:
  - Suppose that  $P(1)$  holds and  $P(k) \rightarrow P(k + 1)$  is true for all positive integers  $k$ .
  - Assume there is at least one positive integer  $n$  for which  $P(n)$  is false. Then the set  $S$  of positive integers for which  $P(n)$  is false is nonempty.
  - By the well-ordering property,  $S$  has a least element, say  $m$ .
  - We know that  $m$  can not be 1 since  $P(1)$  holds.
  - Since  $m$  is positive and greater than 1,  $m - 1$  must be a positive integer. Since  $m - 1 < m$ , it is not in  $S$ , so  $P(m - 1)$  must be true.
  - But then, since the conditional  $P(k) \rightarrow P(k + 1)$  for every positive integer  $k$  holds,  $P(m)$  must also be true. This contradicts  $P(m)$  being false.
  - Hence,  $P(n)$  must be true for every positive integer  $n$ .

# Remembering How Mathematical Induction Works

Consider an infinite sequence of dominoes, labeled  $1, 2, 3, \dots$ , where each domino is standing.

Let  $P(n)$  be the proposition that the  $n$ th domino is knocked over.



We know that the first domino is knocked down, i.e.,  $P(1)$  is true .

We also know that if whenever the  $k$ th domino is knocked over, it knocks over the  $(k + 1)$ st domino, i.e,  $P(k) \rightarrow P(k + 1)$  is true for all positive integers  $k$ .

Hence, all dominos are knocked over.

$P(n)$  is true for all positive integers  $n$ .

# Proving a Summation Formula by Mathematical Induction

**Example:** Show that:  $\sum_{i=1}^n = \frac{n(n+1)}{2}$

**Solution:**

Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

- BASIS STEP:  $P(1)$  is true since  $1(1+1)/2 = 1$ .
- INDUCTIVE STEP: Assume true for  $P(k)$ .

The inductive hypothesis is  $\sum_{i=1}^k = \frac{k(k+1)}{2}$

Under this assumption,

$$\begin{aligned} 1 + 2 + \dots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$





# Conjecturing and Proving Correct a Summation Formula

**Example:** Conjecture and prove correct a formula for the sum of the first  $n$  positive odd integers. Then prove your conjecture.

**Solution:** We have:  $1 = 1$ ,  $1 + 3 = 4$ ,  $1 + 3 + 5 = 9$ ,  $1 + 3 + 5 + 7 = 16$ ,  $1 + 3 + 5 + 7 + 9 = 25$ .

- We can conjecture that the sum of the first  $n$  positive odd integers is  $n^2$ ,

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2.$$

- We prove the conjecture is proved correct with mathematical induction.
- BASIS STEP:  $P(1)$  is true since  $1^2 = 1$ .
- INDUCTIVE STEP:  $P(k) \rightarrow P(k + 1)$  for every positive integer  $k$ .  
Assume the inductive hypothesis holds and then show that  $P(k)$  holds as well.

**Inductive Hypothesis:**  $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

- So, assuming  $P(k)$ , it follows that:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\ &= k^2 + (2k + 1) \quad (\text{by the inductive hypothesis}) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

- Hence, we have shown that  $P(k + 1)$  follows from  $P(k)$ . Therefore the sum of the first  $n$  positive odd integers is  $n^2$ .



# Proving Inequalities

**Example:** Use mathematical induction to prove that  $n < 2^n$  for all positive integers  $n$ .

**Solution:** Let  $P(n)$  be the proposition that  $n < 2^n$ .

- BASIS STEP:  $P(1)$  is true since  $1 < 2^1 = 2$ .
- INDUCTIVE STEP: Assume  $P(k)$  holds, i.e.,  $k < 2^k$ , for an arbitrary positive integer  $k$ .
- Must show that  $P(k + 1)$  holds. Since by the inductive hypothesis,  $k < 2^k$ , it follows that:

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore  $n < 2^n$  holds for all positive integers  $n$ . ◀

# Proving Inequalities

**Example:** Use mathematical induction to prove that  $2^n < n!$ , for every integer  $n \geq 4$ .

**Solution:** Let  $P(n)$  be the proposition that  $2^n < n!$ .

- BASIS STEP:  $P(4)$  is true since  $2^4 = 16 < 4! = 24$ .
- INDUCTIVE STEP: Assume  $P(k)$  holds, i.e.,  $2^k < k!$  for an arbitrary integer  $k \geq 4$ . To show that  $P(k + 1)$  holds:

$$\begin{aligned}2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! && \text{(by the inductive hypothesis)} \\ &< (k + 1)k! \\ &= (k + 1)!\end{aligned}$$

Therefore,  $2^n < n!$  holds, for every integer  $n \geq 4$ . ◀

Note that here the basis step is  $P(4)$ , since  $P(0)$ ,  $P(1)$ ,  $P(2)$ , and  $P(3)$  are all false.

# Proving Divisibility Results

**Example:** Use mathematical induction to prove that  $n^3 - n$  is divisible by 3, for every positive integer  $n$ .

**Solution:** Let  $P(n)$  be the proposition that  $n^3 - n$  is divisible by 3.

- BASIS STEP:  $P(1)$  is true since  $1^3 - 1 = 0$ , which is divisible by 3.
- INDUCTIVE STEP: Assume  $P(k)$  holds, i.e.,  $k^3 - k$  is divisible by 3, for an arbitrary positive integer  $k$ . To show that  $P(k + 1)$  follows:

$$\begin{aligned}(k + 1)^3 - (k + 1) &= (k^3 + 3k^2 + 3k + 1) - (k + 1) \\ &= (k^3 - k) + 3(k^2 + k)\end{aligned}$$

By the inductive hypothesis, the first term  $(k^3 - k)$  is divisible by 3 and the second term is divisible by 3 since it is an integer multiplied by 3. So by part (i) of Theorem 1 in Section 4.1,  $(k + 1)^3 - (k + 1)$  is divisible by 3.

Therefore,  $n^3 - n$  is divisible by 3, for every integer positive integer  $n$ . ◀

# Number of Subsets of a Finite Set

**Example:** Use mathematical induction to show that if  $S$  is a finite set with  $n$  elements, where  $n$  is a nonnegative integer, then  $S$  has  $2^n$  subsets.

*(Chapter 6 uses combinatorial methods to prove this result.)*

**Solution:** Let  $P(n)$  be the proposition that a set with  $n$  elements has  $2^n$  subsets.

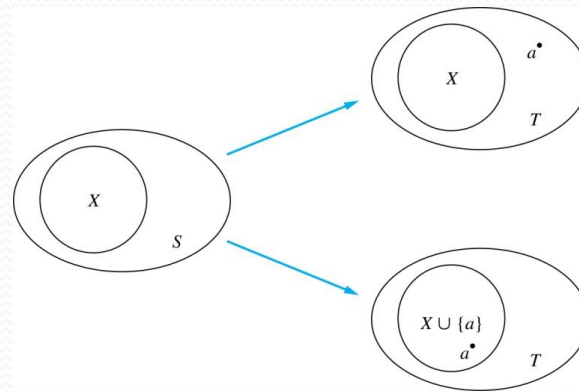
- **Basis Step:**  $P(0)$  is true, because the empty set has only itself as a subset and  $2^0 = 1$ .
- **Inductive Step:** Assume  $P(k)$  is true for an arbitrary nonnegative integer  $k$ .

*continued* →

# Number of Subsets of a Finite Set

**Inductive Hypothesis:** For an arbitrary nonnegative integer  $k$ , every set with  $k$  elements has  $2^k$  subsets.

- Let  $T$  be a set with  $k + 1$  elements. Then  $T = S \cup \{a\}$ , where  $a \in T$  and  $S = T - \{a\}$ . Hence  $|S| = k$ .
- For each subset  $X$  of  $S$ , there are exactly two subsets of  $T$ , i.e.,  $X$  and  $X \cup \{a\}$ .



- By the inductive hypothesis  $S$  has  $2^k$  subsets. Since there are two subsets of  $T$  for each subset of  $S$ , the number of subsets of  $T$  is  $2 \cdot 2^k = 2^{k+1}$ .



# Guidelines: Mathematical Induction Proofs

## *Template for Proofs by Mathematical Induction*

1. Express the statement that is to be proved in the form “for all  $n \geq b$ ,  $P(n)$ ” for a fixed integer  $b$ .
2. Write out the words “Basis Step.” Then show that  $P(b)$  is true, taking care that the correct value of  $b$  is used. This completes the first part of the proof.
3. Write out the words “Inductive Step.”
4. State, and clearly identify, the inductive hypothesis, in the form “assume that  $P(k)$  is true for an arbitrary fixed integer  $k \geq b$ .”
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what  $P(k + 1)$  says.
6. Prove the statement  $P(k + 1)$  making use the assumption  $P(k)$ . Be sure that your proof is valid for all integers  $k$  with  $k \geq b$ , taking care that the proof works for small values of  $k$ , including  $k = b$ .
7. Clearly identify the conclusion of the inductive step, such as by saying “this completes the inductive step.”
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction,  $P(n)$  is true for all integers  $n$  with  $n \geq b$ .